General relativistic cosmology with no beginning of time

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Abstract

We find that general relativity can be naturally free of cosmological singularities. Several nonsingular models are currently available that either assume ad hoc matter contents, or are nonsingular only over a sector of solution space of zero measure, or depart drastically from general relativity at high energies. After much uncertainty over whether cosmological inflation could help solve the initial-singularity problem, the prevailing belief today is that general relativistic cosmology, with inflation or without, is endemically singular. This belief was reinforced by recent singularity theorems that take account specifically of inflation. Here, a viable inflationary cosmology is worked out that is naturally free of singularities despite the fact that 1) it uses only classical general relativity, 2) it assumes only the most generic inflationary matter contents, and 3) it is a theory of the chaotic-inflation type. That type of inflation is the most widely accepted today, as it demands the least fine-tuning of initial conditions. It is also shown how, by dropping the usual simplification of minimal coupling between matter and geometry, the null energy condition can be violated and the relevant singularity theorems circumvented.

The powerful singularity theorems of the sixties [1,2] seemed to leave little room for avoiding a catastrophic singularity in the cosmological past, implying a necessary primeval breakdown of the laws of physics. Naturally, questions were then raised about the self-consistency of general relativity as a complete theory of gravity, a philosophical question the answer to which is still far from clear [3].

There have been several successful attempts at building alternative theories of gravity that are nonsingular in the limit of very high energies (see, e.g., [4-6]). It has also been shown that the equations of motion of several scalar-tensor theories of gravity may admit nonsingular *asymptotic* solutions if the scalar functions in the Lagrangian are reverse-engeneered for that specific purpose [7-9]. Not surprisingly, most of the corresponding *full* solutions are either highly unstable (and hence could not be simulated numerically in the literature) or occupy a sector of measure zero in solution space or in parameter space.

Perhaps the most serious prospect of a nonsingular cosmology based on ordinary gravity arose from the realization that cosmological inflation [10] is probably future-eternal and hence perhaps symmetrically past-eternal as well [11-16]. However, this prospect was dimmed considerably by new inflation-specific singularity theorems [17-21] that condemned a large class of inflationary models to be necessarily singular.

In hindsight, the failure of inflation to *guarantee* the avoidance of a cosmological singularity is not surprising. The inflationary expansion itself, no matter how large, should not affect the global topology of spacetime and hence should have no bearing on the existence or non-existence of global singularities. However, inflation *could* potentially play a role in the singularity game by the mere fact that it assumes the Universe to be dominated by scalar matter at very high energies. Such a universe could easily violate the "strong energy condition" which is assumed by several singularity theorems [2,22-24].

As we shall see below, it is the *pre*-inflationary dynamics and energetics of the scalar matter that are likely to help solve the initial-singularity problem. Once it enters the inflationary phase proper ("slow-roll" conditions) the scalar field acquires dynamics and energetics that do seem to make the model fall under the power of most singularity theorems. More specifically, it starts then to obey the "null energy condition."

We now discuss this latter point more quantitatively. Then, we show in a

specific model how this allows an inflationary cosmology to be nonsingular, even though the characteristics of its "inflationary stage" proper may suggest otherwise. Finally, we follow the evolution of this ordinary-gravity, chaotic inflationary scenario starting from the late inflationary stage, when pre-galactic density fluctuations are generated, back through the very brief but crucial pre-inflationary phase, and finally through the cosmological bounce itself. We confirm numerically that the model is a nonsingular cosmology that is stable in solution space. We conclude with some additional comments on the relevant recent literature.

To see how the quasi-de Sitter phase of inflation does not necessarily effect the initial-singularity problem, consider first the Raychaudhuri equation for null geodesics in the absence of torsion and shear [2]:

$$\frac{d\theta}{dv} + \frac{1}{2}\theta^2 = -R_{\mu\nu}N^\mu N^\nu \quad . \tag{1}$$

θ is the divergence of the considered congruence of null geodesics; v is an affine parameter; $R_{\mu\nu}$ is the Ricci curvature tensor; and the N^μ are components of a null 4-vector field that is tangent to a Robertson-Walker spacetime with metric

$$ds^2 = -dt^2 + a^2(t)dS^2 \quad . \tag{2}$$

dS^2 is the metric of a homogeneous and isotropic 3-space. If one chooses the null 4-vector field to be locally $(1, \overrightarrow{n}/a(t))$, where $\overrightarrow{n}$ is a unit 3-vector, one obtains

$$\frac{d\theta}{dv} + \frac{1}{2}\theta^2 = 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}\right) = 2\left(\dot{H} - \frac{k}{a^2}\right) \quad , \tag{3}$$

where $H \equiv \dot{a}/a$ is the expansion rate of the spacetime and $k = +1$, 0 or -1 according to whether the spatial 3-geometry in Eq.(2) is closed, flat or open, respectively. Dots denote derivation with respect to t.

If one assumes general relativity, considers a model that is driven by a scalar field ϕ with potential $V(\phi)$, and makes the usual simplification of minimal coupling between ϕ and the scalar curvature ($\xi = 0$ in Eq.(10)

below), then $\left(\dot{H} - k/a^2\right)$ is automatically negative definite:

$$\dot{H} - \frac{k}{a^2} = -4\pi G\ \dot{\phi}^2 \quad , \tag{4}$$

where G is the gravitational constant. This seemingly unavoidable fact causes enough past convergence through Eq.(3) that past null geodesics are necessarily incomplete, i.e., the cosmology has a global past singularity [17-21]. Note that a de-Sitter spacetime produces zero on the right-hand-side of Eq.(4) only if it is *exactly* de-Sitter, in which case it cannot be part of a consistent cosmology.

The positivity of $-\left(\dot{H} - k/a^2\right)$, which is essentially the sum of the density and the pressure of the model, also implies that the null energy condition

$$R_{\mu\nu}N^\mu N^\nu \geq 0 \tag{5}$$

is satisfied, bringing such a cosmology under the wrath of several singularity theorems [17-21]. We comment further on this and other energetic aspects at the end of this letter and in [25].

The key Eq.(4) above derives from a Lagrangian of the form

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi) \quad , \tag{6}$$

where R is the scalar curvature. However, it is necessary for the renormalizability of the theory that the Lagrangian contain an explicit coupling term between the scalar field and the scalar curvature [26-28]. The correct form of the Lagrangian is thus

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{1}{2}\xi\phi^2 R + \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi) \quad . \tag{7}$$

Note that in our conventions the conformal value of the nonminimal-coupling constant ξ is $-1/6$. Field theoretic considerations do not a priori constrain the value of ξ (For a more complete discussion, see [29].) However, models with $\xi < 0$ are generally too pathological to yield viable inflationary cosmologies [29,30-34]. In contrast, it was shown some time ago that models with $\xi > 0$, and especially those with $\xi >> 1$, lead to well behaved inflationary

cosmologies with several attractive features[30-37], such as the easing of the observational constraints on the constants that characterize $V(\phi)$.

It can be shown (Eqs.(26,28) of [35]) that during the slow-roll phase of chaotic inflation driven by the theory (7), one has

$$\dot{H} \approx -\frac{8H^2}{(1+6\xi)\phi^2} \quad . \tag{8}$$

Hence, for the cosmologically interesting range $\xi > 0$, one has $\dot{H} < 0$, just as in Eq.(4) where the effect of nonminimal coupling was neglected. Thus, from the above discussion of the Raychaudhuri equation, one could presume that the nonminimal-coupling adjustment would have no obvious effect on the singularity aspect of the model. However, as we shall find soon, the *pre*-inflationary stage of the model *is* drastically effected by the nonminimal coupling, allowing e.g. $\dot{H}$ to assume positive values at the earliest stages, and eventually leading to a nonsingular cosmological past.

Let us then analyze the full equations of motion [32-35] without assuming slow-roll conditions. The "energy" and "momentum" equations are, respectively

$$3\left(H^2+\frac{k}{a^2}\right)\left[\frac{1}{8\pi G}+\xi\phi^2\right] = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 6\xi H\phi\dot{\phi} \quad , \tag{9}$$

$$\left(\dot{H}-\frac{k}{a^2}\right)\left[\frac{1}{8\pi G}+\xi\phi^2\right] = -\frac{1}{2}\dot{\phi}^2 + \xi H\phi\dot{\phi} - \xi\dot{\phi}^2 - \xi\phi\ddot{\phi} \quad . \tag{10}$$

The matter field evolves according to

$$\ddot{\phi} + 3H\dot{\phi} - 6\xi\left(\dot{H}+2H^2+\frac{k}{a^2}\right)\phi + \frac{\partial V(\phi)}{\partial\phi} = 0 \quad . \tag{11}$$

At this point, one cannot limit oneself to looking for de Sitter-like asymptotic solutions and, if there are any, conclude that the model must be nonsingular. In fact, many of the candidate nonsingular models in the literature do

have two exponentially expanding asymptotic solutions, but fail to cross from one asymptotic region to the other, or cross over unstable or zero-measure trajectories in phase space [8,9,38,25]. In fact, the present model illustrates well the caution that must be exercised in inferring the existence or non-existence of singularities from either the properties of the inflationary stage alone or those of the asymptotic solutions in isolation. In the present case, for example, $\dot{H} < 0$ during asymptotic slow-roll phases and yet the model can manage to avoid crushing into a singularity by very briefly shifting away from the slow-roll conditions, allowing $\dot{H}$ to turn positive. Again, there clearly is not a causal relationship between the existence of de-Sitter like asymptotic solutions and the avoidance of cosmological singularities. This aspect is not always recognized, although it had become clear during, e.g., the investigation of "pre-big-bang" cosmology [38-40,5].

Let us choose the potential to be of the simple and generic form $V(\phi) = \lambda\phi^4$. We start at the late stage of inflation when the slow-roll conditions hold. Then, the model evolves according to

$$H^2 \sim -B_1 \log a + C_1 \quad , \quad \phi^2 \sim -B_2 \log a + C_2 \quad , \quad a \sim \exp\left(C_3 t - \frac{1}{4} B_1 t^2\right) \quad , \tag{12}$$

where C_1, C_2 and C_3 are constants that are determined by the initial conditions, while B_1 and B_2 are *positive* constants that depend on the coupling parameters λ and ξ:

$$B_2 = \frac{16}{1+6\xi} \quad , \quad B_1 = \frac{\lambda}{3\xi} B_2 \quad . \tag{13}$$

In addition, $\dot{\phi} \sim -B_2 H / 2\phi$ and fluctuations are produced that will eventually result (when the relevant scales cross back into the Hubble radius after the end of inflation) in pre-galactic density perturbations with magnitude [35,36]

$$\frac{\delta\rho}{\rho} \propto \frac{H^2}{\dot{\phi}(1+6\xi)^{1/2}} \quad . \tag{14}$$

As announced above (see Eq.(8)), $\dot{H}$ is negative-definite during this relatively late stage of inflation: $\dot{H} \approx -B_1/2$. As one evolves the model towards the past, the slow-roll solution (12) breaks down and, a priori, $\dot{H}$ can cover a wide range of values of either sign (see Eqs.(9-10)), depending on the choice of parameters and initial conditions. In a separate paper, we explore the many possible detailed behaviors of the model that can result [25]. Our focus here is the possibility of obtaining a complete and naturally nonsingular inflationary model from fairly generic initial conditions. (By this, we mean of course generic *inflationary* initial conditions. The question of whether the latter are themselves probable among all possible initial conditions is still open [43]).

By inspecting Eqs.(9-10) analytically for the behavior of the matter functions (ϕ, $\dot{\phi}$, $\ddot{\phi}$) and the metric functions (a, H, $\dot{H}$), we find that three types of scenarii are possible.

1) $\dot{H}$ remains negative-definite throughout and H diverges at a finite past value of t where $a(t)$ plunges to zero. I.e., one hits a singularity. This behavior is generic in the case of an open spatial geometry ($k = -1$).

2) $\dot{H}$ first becomes positive (again, as we run the model toward the past) but then decreases toward zero asymptotically. In this case, which obtains typically for spatially flat geometries ($k = 0$), $a(t)$ decreases asymptotically toward a finite minimal size. Such solutions are technically nonsingular, but are usually unstable for most combinations of parameters. Moreover, in the real Universe, the condition of *rigorous* spatial flatness has arguably zero measure, so that unlike in lower-energy cosmological problems, the cases of practical interest here are probably only $k = -1$ and $k = +1$.

3) $\dot{H}$ eventually becomes positive, and stays so sufficiently far in the past to allow H to reach zero and turn negative. The scale factor $a(t)$ reaches a minimum and re-expands again toward the past. This cosmological bounce obtains for the spatially closed case ($k = +1$).

In Figs.(1-3), we confirm numerically the above analysis, focusing on the latter nonsingular case. (More output from the first two cases can be found in [25].) We simulate that scenario starting from the late inflationary stage, back through the change of signs in $\dot{H}$, and finally, safely through the cosmological bounce itself. Fig.(1) shows the evolution of the scalar field ϕ in the neighborhood of the bounce. Fig.(2) displays the overall behavior of the expansion rate H, including the inflationary phases (straight portions of the curve, see Eq.(12)) and the non-inflationary, or inter-inflationary phases

where $\dot{H}$ turns positive and the null energy condition is violated. Fig.(3) shows clearly the distinction between inflationary and inter-inflationary evolutions of the scale factor, and also confirms that the model produces enough expansion to be observationally viable. To facilitate the comparison with the published literature, the figures shown here are obtained for $\xi = 1000$ and $\lambda = 0.001$, just as in [32,33,35]. Qualitatively, the results hold for a broad range of parameter values. One can also obtain some potentially interesting variations by adding a mass term to the Lagrangian or by specializing to very small or very large parameter values [25]. Here, our aim is to show that a generic case can be nonsingular.

We have also checked the special case of minimal coupling ($\xi = 0$). One major difference in this case is that the null energy condition can never be violated (see Eqs.(4,5)), so that most of the singularity theorems do apply. The sub-cases $k = 0$ and $k = -1$ are always singular, as the theorems predict [17-21]. For $k = +1$, the situation is less trivial, because technically the equations of motion do not exclude a bounce. (Note that the inflation-specific singularity theorems are also more technically involved in this case.) For instance, one can set $H = \dot{\phi} = 0$ with $\xi = 0$ and $k = 1$ in Eqs.(9-11), and successfully evolve the system out of that state of minimal size in both directions of time. Unfortunately, because of Eq.(4), H turns negative almost immediately, and the model recollapses far too soon to constitute a viable inflationary scenario. One could still obtain a longer period of expansion by forcing $\dot{\phi}$ to remain virtually zero for a long enough time. For example, this could be arranged by using a potential that is extremely flat near $\phi = 0$ or that has a local minimum around that value, and placing the initial ϕ and $\dot{\phi}$ very close to zero. But one would then be resorting to an inflation of the "new" or the "old" type, respectively. Compared to chaotic inflation, those two types have many additional observational and conceptual difficulties and are considered today to be among the least attractive [10,41-43].

Still on the topic of minimal coupling, a very recent work was just brought to our attention which explores the energy constraints that an inflationary model must satisfy in order to allow for a cosmological bounce [44]. Unfortunately, that study misses the possibility (probably the necessity [26-28]) of nonminimal coupling (e.g., stating inaccurately that all inflationary models are minimally coupled.) The study also concludes that only "old" and "new" inflation ("and not chaotic or eternal inflation") are compatible with the bounce scenario from an energetics point-of-view, which is also inaccurate

as we saw above. Finally, reference [44] does not actually work out a specific inflationary model. Doing so would be necessary because, as we indicated above, one can easily construct solutions that look like they could bounce, either because of their asymptotic behavior or on energetics grounds, but that are not viable cosmologies [7-9,38,25]. Still, [44] is an interesting extension of previous investigations of the interplay between the energy conditions, the singularity theorems, and inflation.

Our results suggest the possibility of a Universe that inflates, then reheats (when ϕ drops close enough to zero) and expands as a Friedmann universe, then recontracts and reheats again, becomes eventually scalar-field dominated, then bounces when it reaches a relatively large minimum size, then inflates and reheats again, and so on for ever in both directions of time (see Figs.(2,3)). Of course, one would have to contend then with the rapid growth of anisotropies and inhomogeneities during contracting phases, and with whether black hole formation can be kept in check, e.g., by the possible dominance of some type of dark matter, in particular some weakly-interacting, nonbaryonic dark matter. Alternatively, it is perhaps possible to cast these findings in the context of “eternal” inflation [11-16], and ask whether the latter can thus be made eternal in the past as well as in the future [21].

We hope to have shown that it might not be necessary to depart much from general relativity to avoid cosmological singularities, were they to be eventually deemed physically or philosophically unacceptable. If, as it is suggested here, the minimal size to which the Universe recollapses is not too small, it might not be necessary to call upon quantum gravity in order to build fully consistent cosmologies.

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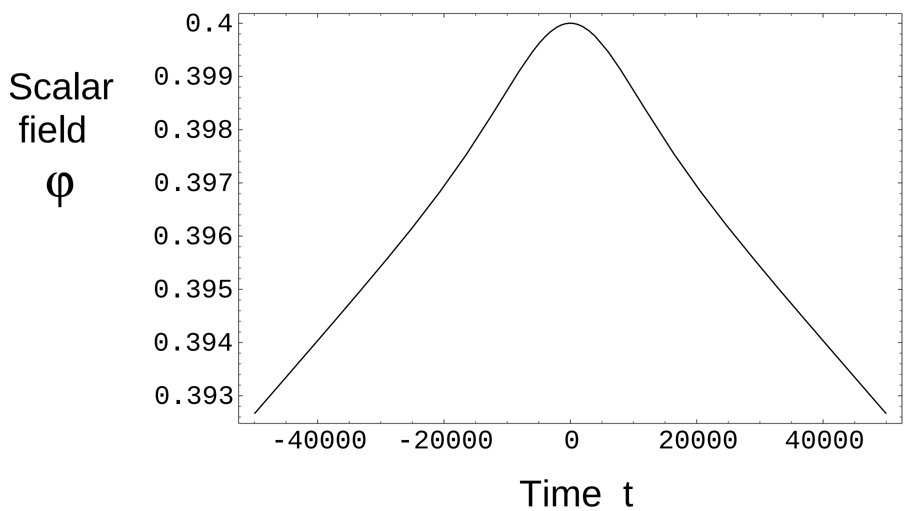
Figure 1



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0.396
0.395
0.394
0.393
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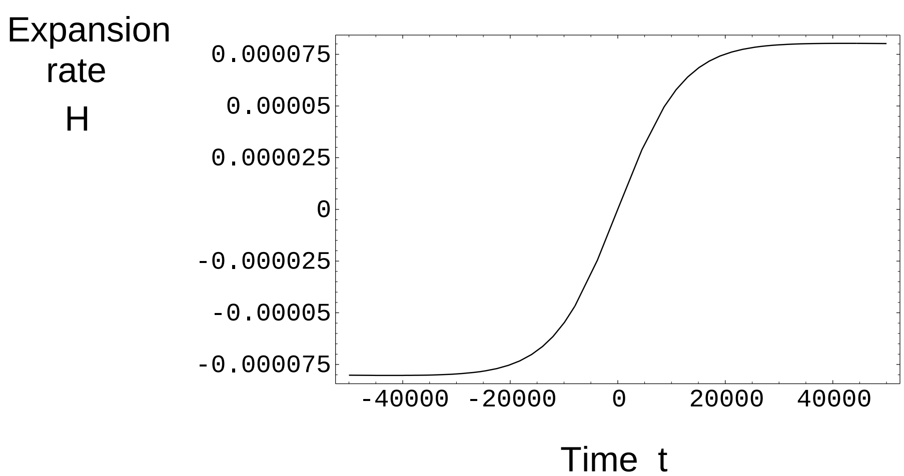
-20000
0
20000
40000
Time t

Figure 2



Expansion
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0
-0.000025
-0.00005
-0.000075
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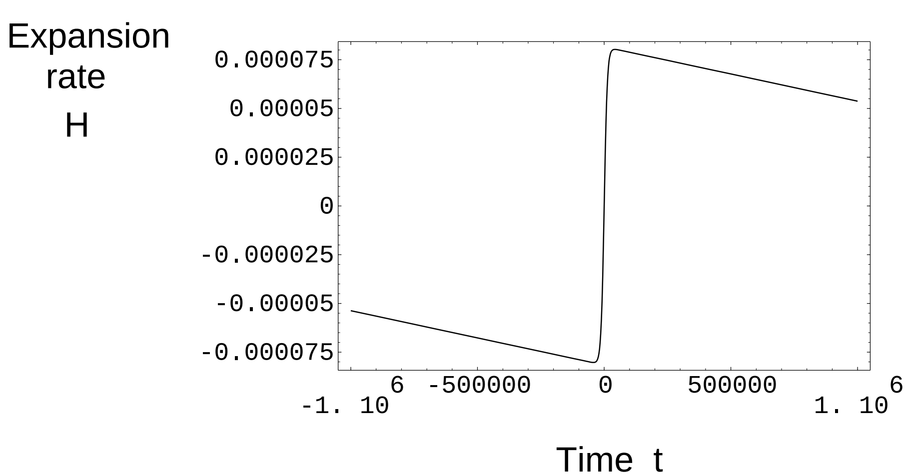
-20000
0
20000
40000
Time t



Expansion
rate
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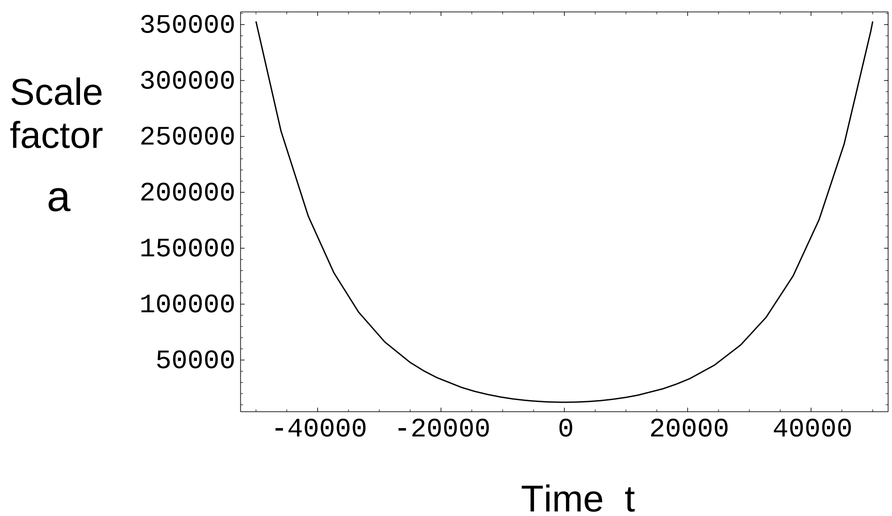
0
500000
1. 10^6
Time t

Figure 3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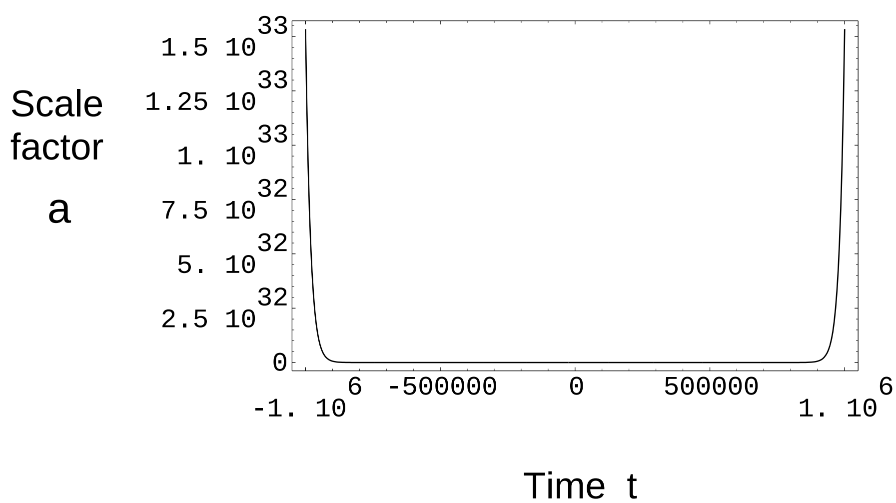
Time t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-500000
0
500000
1. 10^6
Time t